Lecture 13: The Global Diffusion Equation

CBE 30235: Introduction to Nuclear Engineering — D. T. Leighton

February 11, 2026

Reading Assignment

Lamarsh & Baratta (4th Edition):

- **Chapter 5:** Neutron Diffusion.
  - Sections 5.1 – 5.3: Fick's Law and the Equation of Continuity.
- **Chapter 6:** Nuclear Reactor Theory.
  - Section 6.1: One-Group Reactor Equation.

---

1 Introduction: From Micro to Macro

In Lecture 12, we looked at the reactor with a microscope. We examined the "Unit Cell" (a single fuel rod and its water) and defined the **Infinite Multiplication Factor (k_∞)**. This told us whether the fuel/lattice *mixture* was capable of sustaining a chain reaction in an infinite universe.

But reactors are not infinite. They have boundaries. Neutrons leak out. In this lecture, we zoom out to the macroscopic scale. We will derive the equation that balances **Production**, **Absorption**, and **Leakage** to determine if a real, finite reactor can be critical.

2 The Homogenization Approximation

A real reactor core is a complex forest of thousands of fuel rods, control rods, and instrument tubes. Solving the neutron path for every single rod is impossible (by hand) and computationally expensive.

Instead, we use the **Homogenization Approximation**:

- We treat the core as a uniform "smeared" mixture (like a jelly).
- We replace the discrete fuel and water with **Effective Macroscopic Cross-Sections** (Σ_{eff}) and an **Effective Diffusion Coefficient** (D).

Why is this valid? Neutrons bounce around thousands of times before being absorbed. The **Transport Mean Free Path** (λ_{tr}) is typically centimeters, while the reactor is meters wide. Since Reactor Size $\gg \lambda_{tr}$, the neutron sees the "average" properties of the lattice, much like a gas molecule sees average pressure rather than individual collisions.

3 Fick's Law for Neutrons

How do neutrons move through this "smeared" medium? We model their flow using **Diffusion Theory**.

3.1 Neutron Flux (ϕ) vs. Neutron Current (J)

It is critical to distinguish between the *scalar intensity* of the population and the *vector flow* of that population.

- **Flux ($\phi = nv$):** The scalar intensity. It represents the total track length traveled by all neutrons in a unit volume per second.
  - *Physical Meaning:* "How active is the swarm?" (Determines Reaction Rates).
  - *Units:* $\frac{\text{neutrons}}{\text{cm}^2\text{s}}$.
- **Current (J):** The vector net flow. It represents the net number of neutrons crossing a unit area per second.
  - *Physical Meaning:* "Where is the swarm moving?" (Determines Leakage).
  - *Units:* $\frac{\text{neutrons}}{\text{cm}^2\text{s}}$ (Same units, different vector nature!).

3.2 The Notation Trap: A Warning for Chemical Engineers

WARNING: Dimensional Inconsistency Ahead
In standard Chemical Engineering (Transport Phenomena), you are trained to define Fick's Law using **Concentration (n)** and a Diffusivity ($\mathcal{D}_{chem}$) with units of **cm^2/s**. This makes sense because chemical reaction rates depend on concentration.

Nuclear Engineering does it differently. Nuclear reaction rates depend on **Flux** ($\phi = nv$), not just concentration. Because we bake the velocity (v) into the Flux variable, we must remove it from the Diffusion Coefficient to keep the units consistent.

$$J_{nuc} = -D\nabla\phi$$

The Nuclear Diffusion Coefficient D has units of **Length (cm)**, not cm^2/s.

- **Why?** D represents a physical distance: roughly the distance a neutron travels before forgetting its initial direction ($\lambda_{tr}/3$).
- **Benefit:** This removes the wildly varying neutron velocity (v) from the coefficient, making D much more stable across different energy groups.

3.3 Calculating the Nuclear D

From kinetic theory (assuming low absorption), the Nuclear Diffusion Coefficient is determined by the scattering mean free path:

$$D \approx \frac{\lambda_{tr}}{3} = \frac{1}{3\Sigma_{tr}} \quad \text{(units: cm)} \tag{1}$$

For water, $D \approx 0.16$ cm. For graphite, $D \approx 0.85$ cm.

4 The Equation of Continuity

Consider a small control volume V in the reactor. For the number of neutrons n to remain steady (steady-state operation), the neutrons entering, leaving, being born, and dying must balance.

Rate of Change = Production − Absorption − Leakage

Mathematically:

$$\frac{\partial n}{\partial t} = S - \Sigma_a \phi - \nabla \cdot \mathbf{J} \tag{2}$$

For a steady-state reactor ($\frac{\partial n}{\partial t} = 0$), and substituting Fick's Law ($\mathbf{J} = -D\nabla\phi$):

$$D\nabla^2\phi - \Sigma_a\phi + S = 0 \tag{3}$$

5 The One-Group Reactor Equation

Now we specify the source term S. In a nuclear reactor, the source of new neutrons is **Fission**.

- Neutrons are absorbed at a rate $\Sigma_a\phi$.
- For every neutron absorbed, the lattice produces k_∞ new neutrons.

Thus, the source is $S = k_\infty \Sigma_a \phi$. Substituting this back into the diffusion equation and rearranging gives the **One-Group Reactor Equation**:

$$\nabla^2\phi + B_m^2\phi = 0 \tag{4}$$

Where B_m^2 is the **Material Buckling**:

$$B_m^2 \equiv \frac{k_\infty - 1}{L^2} \quad \text{where} \quad L^2 = \frac{D}{\Sigma_a} \tag{5}$$

6 Case Study: Chicago Pile-1 (The First Critical Reactor)

While we use computers to calculate D and k_∞ today, Enrico Fermi had to measure them experimentally. The construction of **CP-1** (Chicago Pile-1) in December 1942 is a perfect application of the diffusion theory we just derived.

6.1 The Lattice Design

Fermi built the reactor under the squash courts of Stagg Field at the University of Chicago.

- **Moderator:** 45,000 Graphite bricks (exceptionally pure to minimize absorption).
- **Fuel:** 40 tons of Uranium Oxide (U_3O_8) and 6 tons of Uranium Metal (the "Spedding Eggs").
- **Geometry:** A rough sphere (eventually flattened to a doorknob shape) to minimize surface area and neutron leakage.

6.2 The Approach to Criticality (Adding Layers)

Fermi did not just build the whole pile and hope it didn't explode. He used a technique called the **"Approach to Critical."**

1. He added one layer of graphite/uranium at a time.
2. After each layer, he lowered a neutron source into the pile and measured the steady-state flux (ϕ).
3. As the pile grew, $k_{effective}$ increased, approaching 1.0.
4. He plotted **1/Count Rate** (or $1/M$) versus the number of layers.

The Physics: As the reactor gets closer to Critical ($k \to 1$), the multiplication M goes to infinity ($M = \frac{1}{1-k}$). Therefore, $1/M$ goes to zero. By extrapolating the line to zero, Fermi could predict exactly which layer would make the reactor critical.

6.3 Criticality: December 2, 1942

On the morning of Dec 2nd, Fermi extrapolated his plot and determined that the **57th layer** would achieve criticality.

- At 3:25 PM, the final control rod was withdrawn by inches.
- The neutron counters began to click exponentially faster without leveling off.
- The reactor was critical ($k \approx 1.0006$). Power was generated at roughly 0.5 Watts.

6.4 Safety Systems (The "Bucket" and the "Axe")

The reactor had no radiation shielding and no cooling system. It relied on three crude but effective safety mechanisms to stop the reaction (make $k < 1$):

1. **The Control Rods:** Cadmium strips (a strong neutron absorber) that could be moved in and out.
2. **ZIP (The Origin of SCRAM?):** A weighted safety rod tied to a rope at the balcony railing. A physicist (Norman Hilberry) stood ready with an axe to cut the rope if the automatic systems failed.
   - *Legend:* "SCRAM" stands for **S**afety **C**ontrol **R**od **A**xe **M**an. (Note: While popular, this acronym is likely a backronym invented later, but the axe man was real!)
3. **The "Suicide Squad":** Three graduate students stood on top of the pile with buckets of **Cadmium Sulfate solution**. If the rods failed to drop, they were ordered to dump the poison directly onto the graphite, destroying the reactor but stopping the chain reaction.

For a detailed history, see: E. Fermi, "Experimental Production of a Divergent Chain Reaction," *American Journal of Physics*, 20, 536 (1952). Also see: Wikipedia: Chicago Pile-1